

SEPARABLE CONVEX OPTIMIZATION PROBLEMS WITH LINEAR ASCENDING CONSTRAINTS *

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Abstract. Separable convex optimization problems with linear ascending inequality and equality constraints are addressed in this paper. Under an ordering condition on the slopes of the functions at the origin, an algorithm that determines the optimum point in a finite number of steps is described. The optimum value is shown to be monotone with respect to a partial order on the constraint parameters. Moreover, the optimum value is convex with respect to these parameters. Examples motivated by optimizations for communication systems are used to illustrate the algorithm.

Key words. ascending constraints, convex optimization, linear constraints, separable problem.

AMS subject classifications. 90C25, 52A41

1. Problem description. Let $g_m, m = 1, 2, \dots, L$ be functions that satisfy the following:

- $g_m : (a_m, b_m) \rightarrow \mathbb{R}$ where $a_m \in [-\infty, 0)$ and $b_m \in (0, +\infty]$ and therefore $a_m < 0 < b_m$;
- g_m is strictly convex in its domain (a_m, b_m) ;
- g_m is continuously differentiable in its domain (a_m, b_m) ;
- The slopes of the functions at 0, i.e., the values of the strictly increasing function $h_m := g'_m$ at 0, are in increasing order with respect to the index m :

$$h_1(0) \leq h_2(0) \leq \dots \leq h_L(0); \quad (1.1)$$

- There is a point in the domain (a_m, b_m) where the slope of g_m equals $h_1(0)$, the slope of the first function at 0. (This may be equivalently stated as $h_1(0) \geq h_m(a_m+)$, given (1.1) and that h_m is continuous and strictly increasing).

In this paper, we minimize the separable objective function $G : \mathbb{R}^L \rightarrow \mathbb{R}$ given by

$$G(y) := \sum_{m=1}^L g_m(y_m), \quad (1.2)$$

where $y = (y_1, \dots, y_L)$, subject to the following linear inequality and equality constraints:

$$y_m \in [0, \beta_m], \quad m = 1, 2, \dots, L, \quad (1.3)$$

$$\sum_{m=1}^l y_m \geq \sum_{m=1}^l \alpha_m, \quad l = 1, 2, \dots, L-1, \quad (1.4)$$

$$\sum_{m=1}^L y_m = \sum_{m=1}^K \alpha_m. \quad (1.5)$$

*This work was supported by the University Grants Commission under Grant Part (2B) UGC-CAS-(Ph.IV) and by the Department of Science and Technology under Grant DSTO748.

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In the above constraints, we assume $\beta_m \in (0, b_m]$ for $m = 1, 2, \dots, L$, $\alpha_m \geq 0$ for $m = 1, 2, \dots, K$, where $K \geq L$, and naturally,

$$\sum_{m=1}^K \alpha_m \leq \sum_{m=1}^L \beta_m. \quad (1.6)$$

We also assume

$$\sum_{m=L}^K \alpha_m > 0. \quad (1.7)$$

The inequalities in (1.3) impose positivity and upper bound constraints. Note that if $\beta_m = b_m$, the upper bound constraint is irrelevant because the domain of g_m is (a_m, b_m) . The inequalities in (1.4) impose a sequence of *ascending constraints* with increasing heights $\sum_{m=1}^l \alpha_m$ indexed by l . Assumption (1.6) is necessary for the constraint set to be nonempty. Without (1.7), it is easy to see that $y_L = 0$, and the problem reduces to a similar one with fewer variables.

What we have described is a separable convex optimization problem with linear inequality and equality constraints. A rich duality theory exists for such problems. See Bertsekas [1, Sec. 5.1.6]. Here, we provide an algorithm that puts out a vector that minimizes (1.2) and terminates in at most L steps. Section 2 contains a description of the algorithm and Section 4 the proof of its optimality. While we may take $K = L$ without loss of generality, allowing $K \geq L$ simplifies the exposition of our algorithm.

Problems of the above kind arise in the optimization of multi-terminal communication systems where power utilized, measured in Joules per second, is minimized, or throughput achieved, measured in bits per second, is maximized, subject to meeting certain quality of service and feasibility constraints. See Viswanath & Anantharam [2] for details and Section 3 for specific examples. Viswanath & Anantharam [2] provide two algorithms for their power minimization and throughput maximization problems. Our work unites their solutions and goes further to minimize any G that satisfies the constraints mentioned above. Under a further condition on the functions which will be stated in Section 2, we argue that our algorithm provides the solution to the above optimization problem with the additional ordering constraint $y_1 \geq y_2 \geq \dots \geq y_L$.

2. The Main Results.

We begin with some remarks on notation.

- For integers i, j satisfying $i \leq j$, we let $\llbracket i, j \rrbracket$ denote the set $\{i, i+1, \dots, j\}$.
- Let $\mathbb{E}_m := h_m((a_m, b_m))$, the range of h_m . Thus the condition $h_1(0) > h_m(a_m+), m \in \llbracket 1, L \rrbracket$ in Section 1 may be written as

$$h_1(0) \in \cap_{m=1}^L \mathbb{E}_m. \quad (2.1)$$

- Denote by $h_m^{-1} : \mathbb{E}_m \rightarrow (a_m, b_m)$ the inverse of the continuous and strictly increasing function h_m . The inverse is also continuous and strictly increasing in its domain.
- For convenience, define the functions $H_m : \mathbb{E}_m \rightarrow (a_m, \beta_m]$ to be

$$H_m := h_m^{-1} \wedge \beta_m. \quad (2.2)$$

H_m is clearly increasing.¹ Assignments to the variable y_m will be via evaluation of H_m so that the upper bound constraint in (1.3) is automatically satisfied.

¹We say f is increasing if $a > b$ implies $f(a) \geq f(b)$. If there is strict inequality, we say f is strictly increasing. Similarly we use positive for ≥ 0 and strictly positive for > 0 .

- For $1 \leq i \leq l < L$, let θ_i^l denote the least $\theta \geq h_1(0)$ that satisfies the equation

$$\sum_{m=i}^l H_m(\theta) = \sum_{m=i}^l \alpha_m, \quad (2.3)$$

provided the set of such θ is nonempty. Otherwise we say θ_i^l does not exist. The domain of $\sum_{m=i}^l H_m$ is $\cap_{m=i}^l \mathbb{E}_m$. The function $\sum_{m=i}^l H_m$ is increasing, and moreover, strictly increasing until all functions in the sum saturate. So there is no solution to (2.3) when for example $\sum_{m=i}^l \alpha_m > \sum_{m=i}^l \beta_m$. In general, if we can demonstrate the existence of $\underline{\theta}$ and $\bar{\theta}$, both in the set $\cap_{m=i}^l \mathbb{E}_m$, that satisfy

$$\sum_{m=i}^l H_m(\underline{\theta}) \leq \sum_{m=i}^l \alpha_m \leq \sum_{m=i}^l H_m(\bar{\theta}), \quad (2.4)$$

then the existence of $\theta_i^l \in \cap_{m=i}^l \mathbb{E}_m$ is assured, thanks to the continuity of $\sum_{m=i}^l H_m$. Indeed, we may always take $\underline{\theta} = h_1(0)$. This is because our assumptions (2.1), (1.1), and the increasing property of $H_m, m \in \llbracket 1, L \rrbracket$ imply

$$\sum_{m=i}^l H_m(h_1(0)) \leq \sum_{m=i}^l H_m(h_m(0)) = \sum_{m=i}^l (h_m^{-1}(h_m(0)) \wedge \beta_m) = 0 \leq \sum_{m=i}^l \alpha_m.$$

The continuity and increasing property of $\sum_{m=i}^l H_m$ further imply that

$$(\underline{\theta} \wedge h_1(0)) \leq \theta_i^l \leq \bar{\theta}. \quad (2.5)$$

Thus, in order to show existence of θ_i^l , it is sufficient to identify a $\bar{\theta}$ that satisfies the right side inequality of (2.4). We will have occasion to use this remark a few times in the proof of correctness of the algorithm.

- Similarly, for $1 \leq i \leq j \leq L$, we let Θ_i^j denote the least $\theta \geq h_1(0)$ that satisfies the equation

$$\sum_{m=i}^j H_m(\theta) = \sum_{m=i}^K \alpha_m, \quad (2.6)$$

provided the set of such θ is nonempty. Otherwise we say Θ_i^j does not exist. The difference between (2.3) and (2.6) is the summation up to K in the right side of (2.6) and the consequent difference in the upper limits on the left and right sides of (2.6). Hence the upper case Θ_i^j . The remarks made above on the existence of θ_i^l are applicable to Θ_i^j .

- We now provide a description of the variables used in the algorithm for ease of reference.
 - n : Iteration number.
 - i_n and j_n : Pointer locations of the first and the last variables, y_{i_n} and y_{j_n} , that are yet to be set.
 - N : The last iteration number in which a variable is set.
 - l, m : Temporary pointer locations that satisfy $l, m \in \llbracket i_n, j_n \rrbracket$ in iteration n .

- t : Pointer to the variable that satisfies the corresponding ascending constraint with equality; $t \in \llbracket i_n, j_n \rrbracket$.
- ξ_n : Choice of the best slope (marginal cost) in iteration n .
- p_m : Iteration number when variable y_m is set. (This is needed only in the proof).
- c_m : A label that indicates the type of ξ_n that set the variable y_m . The possible labels are $\{\mathcal{A}, \mathcal{A}^*\}$ for Step 3(a) of the algorithm, $\{\mathcal{B}^*\}$ for Step 3(b) of the algorithm, and $\{\mathcal{C}, \mathcal{C}^*\}$ for Step 3(c) of the algorithm. If c_m is assigned an asterisked label, then the ascending constraint is met with equality for y_m . (This is needed only in the proof).

We now provide a generalization of the two algorithms given by Viswanath & Anantharam [2].

ALGORITHM 1.

- **Inputs:** $K, L, (\alpha_1, \alpha_2, \dots, \alpha_K), (\beta_1, \beta_2, \dots, \beta_L)$.
- **Output:** $y^* = (y_1^*, y_2^*, \dots, y_L^*)$.
- **Step 1: Initialization** Set $n \leftarrow 1, i_1 \leftarrow 1, j_1 \leftarrow L$ and go to **Step 2**.
- **Step 2: Termination** If $i_n > j_n$, then set $N \leftarrow n - 1$, output the vector $y^* = (y_1^*, y_2^*, \dots, y_L^*)$, and **stop**.
Else go to **Step 3**.
- **Step 3:** Find $\Theta_{i_n}^{j_n}$, the solution of (2.6) with $i \leftarrow i_n$ and $j \leftarrow j_n$. Also find $\theta_{i_n}^l$ for $l \in \llbracket i_n, j_n - 1 \rrbracket$. These are solutions of (2.3) for $i \leftarrow i_n$ and l as chosen.² Then set

$$\xi_n = \max \left\{ \Theta_{i_n}^{j_n}, h_{j_n}(0), \theta_{i_n}^l; l \in \llbracket i_n, j_n - 1 \rrbracket \right\}$$

- **Case 3(a):** If $\xi_n = \Theta_{i_n}^{j_n}$, then set

$$\begin{aligned} y_m^* &\leftarrow H_m(\xi_n) \text{ for } m \in \llbracket i_n, j_n \rrbracket \\ p_m &\leftarrow n \text{ for } m \in \llbracket i_n, j_n \rrbracket \\ c_m &\leftarrow \mathcal{A} \text{ for } m \in \llbracket i_n, j_n - 1 \rrbracket \\ c_{j_n} &\leftarrow \mathcal{A}^* \\ i_{n+1} &\leftarrow j_n + 1 \\ j_{n+1} &\leftarrow j_n. \\ n &\leftarrow n + 1 \end{aligned}$$

Go to **Step 2**.

- **Case 3(b):** If $\xi_n = h_{j_n}(0)$, then set

$$\begin{aligned} y_{j_n}^* &\leftarrow 0 \\ p_{j_n} &\leftarrow n \\ c_{j_n} &\leftarrow \mathcal{B}^* \\ i_{n+1} &\leftarrow i_n \\ j_{n+1} &\leftarrow j_n - 1 \\ n &\leftarrow n + 1. \end{aligned}$$

Go to **Step 2**.

²Theorem 1 gives a sufficient condition when these quantities can be identified in every iteration.

- **Case 3(c):** If $\xi_n = \theta_{i_n}^t$, for $t = \llbracket i_n, j_n - 1 \rrbracket$, pick the largest such t and set

$$\begin{aligned} y_m^* &\leftarrow H_m(\xi_n) \text{ for } m \in \llbracket i_n, t \rrbracket \\ p_m &\leftarrow n \text{ for } m \in \llbracket i_n, t \rrbracket \\ c_m &\leftarrow \mathcal{C} \text{ for } m \in \llbracket i_n, t - 1 \rrbracket \\ c_t &\leftarrow \mathcal{C}^* \\ i_{n+1} &\leftarrow t + 1 \\ j_{n+1} &\leftarrow j_n. \\ n &\leftarrow n + 1. \end{aligned}$$

Go to Step 2. ■

Remarks:

- Observe that in each iteration (i.e., a call to Step 3) at least one variable is set. So the algorithm terminates within L steps.
- The iterations are indexed by n where $n \in \llbracket 1, N \rrbracket$. In each iteration, say n , the pointers i_n and j_n indicate the start and end variables that are yet to be set. At the end of this iteration, either all the variables are set (Step 3(a)), or the last one alone is set to 0 (Step 3(b)), or the variables with contiguous indices $\llbracket i_n, t \rrbracket$ are set (Step 3(c)). The corresponding sets of labels are $\{\mathcal{A}, \mathcal{A}^*\}$, $\{\mathcal{B}^*\}$, and $\{\mathcal{C}, \mathcal{C}^*\}$, respectively.
- Suppose y is the vector of production levels of L production units. Let g_m represent the cost of operation for production unit $m \in \llbracket 1, L \rrbracket$, and G the overall cost. The production levels y_m set in a particular iteration are set to have the same marginal cost ξ_n , or are set to operate at capacity. In symbols, $y_m^* = H_m(\xi_{p_m}) = h_m^{-1}(\xi_{p_m}) \wedge \beta_m$.
- Each iteration requires the evaluation of $\Theta_{i_n}^{j_n}$, and $\theta_{i_n}^l$ for $l \in \llbracket i_n, j_n - 1 \rrbracket$. These are zeros of continuous increasing functions. In Theorem 1 below, we provide sufficient conditions under which these zeros exist in each iteration step.
- The question of evaluation of these zeros naturally arises. In the specific examples in Section 3, we give closed form expressions for $\Theta_{i_n}^{j_n}$ and $\theta_{i_n}^l$. In general, this may not be available and one has to resort to numerical evaluation. However, the observation that the functions are continuous and increasing enables an efficient line search for the zeros. In the proof, we identify $\underline{\theta}$ and $\bar{\theta}$ on either side of the zero (see (2.4) and (2.5)) that narrows the search window.

We now state the main result of the paper.

THEOREM 1. *If $\theta_1^l, l \in \llbracket 1, L - 1 \rrbracket$ and Θ_1^L exist, then the following hold.*

- *For every iteration n with $i_n \leq j_n$, the quantities $\Theta_{i_n}^{j_n}$ and $\theta_{i_n}^l, l \in \llbracket i_n, j_n - 1 \rrbracket$ exist.*
- *Algorithm 1 terminates in $N \leq L$ iterations.*
- *The output of Algorithm 1 minimizes (1.2) under the stated constraints.*

We next state a simple corollary to this result which solves a related problem with additional constraints.

COROLLARY 2.1. *If the functions H_m satisfy $H_1 \geq H_2 \geq \dots \geq H_L$, then under the conditions of Theorem 1, the optimum y^* satisfies $y_1^* \geq y_2^* \geq \dots \geq y_L^*$.*

Remark: We may use Algorithm 1 to solve the minimization problem with the additional constraints $y_1 \geq y_2 \geq \dots \geq y_L$ if H_m is point-wise monotone decreasing in the index m .

Before we state some properties of the optimum value function, we make some more definitions for convenience.

- Observe that if $K > L$, the optimum value defined below depends on the K -tuple $\alpha \in \mathbb{R}_+^K$ only through the L -tuple $(\alpha_1, \dots, \alpha_{L-1}, \sum_{m=L}^K \alpha_m) \in \mathbb{R}_+^L$. For studying the optimum value, we may therefore restrict our attention to $K = L$. Let $\alpha \in \mathbb{R}_+^L$ and define $\mathcal{G} : \mathbb{R}_+^L \rightarrow \mathbb{R} \cup \{+\infty\}$ as follows:

$$\alpha \mapsto \mathcal{G}(\alpha) := \inf \{G(y) : y \in \mathbb{R}^L \text{ satisfies constraints (1.3) – (1.5)}\}$$

We do not place the restrictions (1.6) and (1.7) on α ; if the optimization is over an empty set the infimum is taken to be $+\infty$. Clearly $\mathcal{G} > -\infty$ because it is the infimum of a strictly convex function over a bounded convex set, the set being defined by the constraints (1.3)-(1.5).

- Define a partial order on \mathbb{R}_+^K as follows. We say $\alpha \succeq \tilde{\alpha}$ if

$$\sum_{m=1}^l \alpha_m \geq \sum_{m=1}^l \tilde{\alpha}_m, \quad l = \llbracket 1, L \rrbracket,$$

with equality when $l = L$.

This partial order is stronger than majorization (see for example Marshall & Olkin [3]) in the sense that if $\alpha \succeq \tilde{\alpha}$, then α majorizes $\tilde{\alpha}$. Loosely speaking, $\alpha \succeq \tilde{\alpha}$ indicates that the components for α are lopsided relative to those of $\tilde{\alpha}$. The proposition below says that lopsidedness increases cost.

PROPOSITION 2.2. *The function \mathcal{G} satisfies the following properties:*

- If $\alpha \succeq \tilde{\alpha}$, then $\mathcal{G}(\alpha) \geq \mathcal{G}(\tilde{\alpha})$.
- \mathcal{G} is a convex function.

All the above results are generalizations of those of Viswanath & Anantharam [2]. The proofs are in Section 4.

3. Examples. We first consider a special case of an example from Bertsekas [1, Ex. 5.1.2] that is of interest in optimization of communication systems.

EXAMPLE 1 (Vector Gaussian Channel). *Consider L channels with noise variances $\sigma_m^2, m \in \llbracket 1, L \rrbracket$. If power $y_m (\geq 0)$ is allocated to channel m , the throughput on this channel is $\log \left(1 + \frac{y_m}{\sigma_m^2}\right)$. Maximize the total throughput*

$$\sum_{m=1}^L \log \left(1 + \frac{y_m}{\sigma_m^2}\right) \tag{3.1}$$

subject to a sum power constraint $\sum_{m=1}^L y_m = P$.

The optimal allocation of powers is usually called “water-filling” allocation [4, Sec. 10.4] because it levels the noise-plus-signal power $\sigma_m^2 + y_m^*$ in the channels subject to the sum power constraint, possibly leaving out a few of the noisiest dimensions. We now arrive at this solution using our algorithm.

Without loss of generality, we arrange the indices so that

$$\sigma_1^2 \leq \sigma_2^2 \leq \dots \leq \sigma_L^2. \quad (3.2)$$

Set $g_m(x) = -\log\left(1 + \frac{x}{\sigma_m^2}\right)$, $(a_m, b_m) = (-\sigma_m^2, +\infty)$, $\beta_m = +\infty$, $K = L$, $\alpha_m = 0$ for $m \in \llbracket 1, L-1 \rrbracket$, and $\alpha_L = P$. It is easy to verify that $g_m, m \in \llbracket 1, L \rrbracket$ satisfy all the conditions laid out in Section 1 and that

$$H_m(\theta) = -\theta^{-1} - \sigma_m^2 \quad (3.3)$$

with domain $\mathbb{E}_m = (-\infty, 0)$. Consequently, θ_1^l , the solution to (2.3), is given by

$$\theta_1^l = \frac{-l}{\sum_{m=1}^l \sigma_m^2}, l \in \llbracket 1, L-1 \rrbracket, \quad (3.4)$$

and

$$\Theta_1^L = \frac{-L}{\sum_{m=1}^L \sigma_m^2 + P}. \quad (3.5)$$

Theorem 1 therefore indicates that Algorithm 1 is applicable. θ_1^l and Θ_1^j are similarly identified for $1 \leq l < j \leq L$. The ordering in (3.2) implies that $\Theta_1^j \geq \theta_1^l$ for all $l \in \llbracket 1, j-1 \rrbracket$. An execution of Algorithm 1 therefore results in the following: identify the largest j such that $\Theta_1^j \geq h_j(0) = -\sigma_j^{-2}$, or equivalently, $\sigma_j^2 \leq -1/\Theta_1^j$, set $y_m^* = 0$ for $m \in \llbracket j+1, L \rrbracket$, and set $y_m = H_m(\Theta_1^j)$ for $m \in \llbracket 1, j \rrbracket$. From (3.3), we see that $y_m^* + \sigma_m^2 = -1/\Theta_1^j$, the water level, for $m \in \llbracket 1, j \rrbracket$. ■

Our second example is closely related to Example 1 and is from Viswanath & Anantharam [2]. It evaluates the sum throughput in a multi-user setting. Omitting the details of the reduction, we present only the mathematical abstraction.

EXAMPLE 2 (Sum Capacity). *Let (3.2) hold. Let $g_m(x) = -\log\left(1 + \frac{x}{\sigma_m^2}\right)$, $m \in \llbracket 1, L \rrbracket$, as in Example 1, and let $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_K \geq 0$. Minimize (1.2) subject to the constraints in Section 1.*

The difference between Examples 1 and 2 is that the ascending constraints now apply. Existence of $\theta_1^l, l \in \llbracket 1, L-1 \rrbracket$ and Θ_1^L follows as in the previous example (see (3.6) below), and Algorithm 1 can be used to solve the minimization. Analogous to (3.4) and (3.5), θ_i^l and Θ_i^j are given by

$$\theta_i^l = \frac{-(l-i+1)}{\sum_{m=i}^l \sigma_m^2} \text{ and } \Theta_i^j = \frac{-(j-i+1)}{\sum_{m=i}^j \sigma_m^2 + P}. \quad (3.6)$$

for $1 \leq i \leq l < j \leq L$. At iteration step n , given i_n and j_n , the assignment for ξ_n is

explicitly given by

$$\xi_n = \max \left\{ \frac{-(j_n - i_n + 1)}{\sum_{m=i_n}^{j_n} \sigma_m^2 + \sum_{m=i_n}^K \alpha_m}, \frac{-1}{\sigma_{j_n}^2}, \frac{-(l - i_n + 1)}{\sum_{m=i_n}^l \sigma_m^2 + \sum_{m=i_n}^l \alpha_m}; l \in \llbracket i_n, j_n - 1 \rrbracket \right\}. \quad (3.7)$$

Viswanath & Anantharam [2] give the same condition in terms of $-1/\xi_n$. The optimal allocation at this step sets either $y_{j_n}^* = 0$ indicating a rejection of the noisy channel index j_n , or $y_m^* = H_m(\xi_n)$, i.e., $y_m^* + \sigma_m^2 = -\xi_n^{-1}$ for a contiguous set of channels starting from index i_n . This optimal allocation thus partitions the channels into sets of contiguous channels, with each partition having its own water level.

Remark: Viswanath & Anantharam make an incorrect claim in [2, Appendix A.5] that the specific algorithm with ξ_n set via (3.7) puts out the optimal y^* whenever $g_m, m \in \llbracket 1, L \rrbracket$ are of the form $g_m(x) = f\left(1 + \frac{x}{\sigma_m^2}\right)$ and $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ is a continuous, increasing, strictly concave function. Their proof works only for some special cases. In particular, it works for $f(x) = \log x$ as in Example 2 above. The error in their proof can be traced to an incomplete proof for the case when $L = 2$ in [2, p.1309]; the validity of their statement for $L = 2$ holds only in some special cases, $f(x) = \log(x)$ being one of them. Of course, Algorithm 1 with the correct ξ_n based on the functions $g_m, m \in \llbracket 1, L \rrbracket$ will yield the optimal y^* . ■

Our third example is also taken from Viswanath & Anantharam [2] and evaluates the minimum power required to meet a quality-of-service constraint. It serves to illustrate the use of Corollary 2.1. The mathematical abstraction is as follows.

EXAMPLE 3. *Let (3.2) hold. Let $g_m: (-\infty, 1) \rightarrow \mathbb{R}_+$ be defined as $g_m(x) = \frac{\sigma_m^2}{1-x}, m \in \llbracket 1, L \rrbracket$. Let $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_K$. Minimize (1.2) subject to the constraints in Section 1 and the additional set of constraints $y_1 \geq y_2 \geq \dots \geq y_L$.*

Let us first solve the problem ignoring the constraint $y_1 \geq y_2 \geq \dots \geq y_L$. Observe that $(a_m, b_m) = (-\infty, 1)$ for all m and all the conditions outlined in Section 1 are satisfied by the given set of functions. Furthermore, it is easy to verify that $\mathbb{E}_m = (0, \infty)$ and $H_m(\theta) = 1 - \frac{\sigma_m}{\sqrt{\theta}}$. The quantities θ_1^l and Θ_1^L exist if $\sum_{m=1}^l \alpha_m < l$ and $\sum_{m=1}^K \alpha_m < L$. More generally, for $1 \leq i \leq l < j \leq L$, θ_i^l and Θ_i^j are given by

$$\theta_i^l = \left(\frac{\sum_{m=i}^l \sigma_m}{l - i + 1 - \sum_{m=i}^l \alpha_m} \right)^2$$

and

$$\Theta_i^j = \left(\frac{\sum_{m=i}^j \sigma_m}{j - i + 1 - \sum_{m=i}^j \alpha_m} \right)^2.$$

Theorem 1 then assures us that Algorithm 1 yields the optimum solution. At iteration step n , given i_n and j_n , the assignment for ξ_n can once again be made more explicit and our algorithm reduces to the second algorithm of Viswanath & Anantharam.

Observe now that (3.2) implies that $H_1 \geq H_2 \geq \dots \geq H_L$ so that the optimal y^* put out by Algorithm 1 also satisfies $y_1^* \geq y_2^* \geq \dots \geq y_L^*$. ■

Our final example illustrates the handling of the upper bound constraint. This is another special case of the example from Bertsekas [1, Ex. 5.1.2] that arises in a power optimization problem for sensor networks (see Zacharias & Sundaresan [5]).

EXAMPLE 4. Let $g_m(x) = \frac{1}{2}x^2, m \in \llbracket 1, L \rrbracket, K = L, \alpha_m = 0, m \in \llbracket 1, L-1 \rrbracket, \alpha_L = \alpha > 0$, and $\beta_m \in (0, \infty)$ for $m \in \llbracket 1, L \rrbracket$. Further, order the indices so that $\beta_1 \leq \beta_2 \leq \dots \leq \beta_L$. Minimize (1.2) under this setting.

Once again, all conditions outlined in Section 1 hold. It is easy to verify that $H_m(\theta) = \theta \wedge \beta_m$. The function $\sum_{m=1}^L H_m$ is a piece-wise linear continuous function passing through the origin with slope l in $(-\infty, \beta_1)$, slope $(l-1)$ in (β_1, β_2) , and so on, and zero-slope in $(\beta_L, +\infty)$. Clearly $\theta_1^l = 0, l \in \llbracket 1, L-1 \rrbracket$. We assume that Θ_1^L exists which is equivalent to $\alpha \leq \sum_{m=1}^L \beta_m$. Yet again, Theorem 1 assures us that Algorithm 1 is applicable.

An application of Algorithm 1 results in the following. Identify the unique k such that $\alpha \in [a_k, a_{k+1})$ where

$$a_k := (L-k)\beta_k + \sum_{m=1}^k \beta_m.$$

(It is easy to see that $a_k \leq a_{k+1}$). Then

$$\Theta_1^L = \frac{\alpha - \sum_{m=1}^k \beta_m}{L-k}.$$

Moreover, $y_m^* = H_m(\Theta_1^L) = \beta_m$ for $m \in \llbracket 1, k \rrbracket$. For $m = \llbracket k+1, L \rrbracket$, the values are suitably lowered from their upper bounds. Note that this assignment is completed in just one iteration of Algorithm 1. ■

4. Proofs.

4.1. Preliminaries. We first prove some facts on the individual cases.

LEMMA 4.1. Suppose that in iteration n , the quantities $\theta_{i_n}^l, l \in \llbracket i_n, j_n-1 \rrbracket$ and $\Theta_{i_n}^{j_n}$ exist. Suppose further that $\xi_n = h_{j_n}(0)$ and Step 3(b) is executed. Then the following hold.

- The quantities $\theta_{i_{n+1}}^l, l \in \llbracket i_{n+1}, j_{n+1}-1 \rrbracket$ and $\Theta_{i_{n+1}}^{j_{n+1}}$ exist.
- $\xi_n \geq \xi_{n+1}$.

Proof: Given that $\xi_n = h_{j_n}(0)$ and Step 3(b) is executed, we see that $i_{n+1} = i_n$ and $j_{n+1} = j_n - 1$. We may assume $i_{n+1} \leq j_{n+1}$; otherwise there is nothing to prove. Since the start pointer does not change and the end pointer decrements by 1, it is clear that $\theta_{i_{n+1}}^l = \theta_{i_n}^l$ for $l \in \llbracket i_{n+1}, j_{n+1}-1 \rrbracket$ because $\theta_{i_{n+1}}^l$ and $\theta_{i_n}^l$ are zeros of identical functions for the indicated values of l .

The question of existence now reduces to that of only $\Theta_{i_{n+1}}^{j_{n+1}}$ (recall definition in (2.6)). First observe that if Step 3(b) is executed, we must have $\xi_n = h_{j_n}(0) > \Theta_{i_n}^{j_n}$, and therefore

$$0 = h_{j_n}^{-1}(h_{j_n}(0)) = h_{j_n}^{-1}(\xi_n) \geq H_{j_n}(\xi_n) \geq H_{j_n}(\Theta_{i_n}^{j_n}), \quad (4.1)$$

where the last inequality follows because H_{j_n} is increasing. Consequently, we must have

$$\begin{aligned}
\sum_{m=i_{n+1}}^K \alpha_m &= \sum_{m=i_n}^K \alpha_m = \sum_{m=i_n}^{j_n} H_m \left(\Theta_{i_n}^{j_n} \right) \quad \left(\text{from definition of } \Theta_{i_n}^{j_n} \right) \\
&= \sum_{m=i_n}^{j_n-1} H_m \left(\Theta_{i_n}^{j_n} \right) + H_{j_n} \left(\Theta_{i_n}^{j_n} \right) \\
&\leq \sum_{m=i_n}^{j_n-1} H_m \left(\Theta_{i_n}^{j_n} \right) + 0 \quad \left(\text{from (4.1)} \right) \\
&= \sum_{m=i_{n+1}}^{j_{n+1}} H_m \left(\Theta_{i_n}^{j_n} \right).
\end{aligned}$$

So we may take $\bar{\theta} = \Theta_{i_n}^{j_n}$ in (2.4). On the lower side, we may simply use $\underline{\theta} = h_1(0)$. However, we can find a tighter bound from the following sequence of inequalities

$$\begin{aligned}
\sum_{m=i_{n+1}}^{j_{n+1}} H_m \left(\theta_{i_n}^{j_n-1} \right) &= \sum_{m=i_n}^{j_n-1} H_m \left(\theta_{i_n}^{j_n-1} \right) = \sum_{m=i_n}^{j_n-1} \alpha_m \quad \left(\text{from definition of } \theta_{i_n}^{j_n-1} \right) \\
&< \sum_{m=i_n}^K \alpha_m \quad \left(\text{from (1.7)} \right) \\
&= \sum_{m=i_{n+1}}^K \alpha_m;
\end{aligned}$$

i.e., we may take $\underline{\theta} = \theta_{i_n}^{j_n-1}$ in (2.4). $\Theta_{i_{n+1}}^{j_{n+1}}$ therefore exists and

$$\theta_{i_n}^{j_n-1} < \Theta_{i_{n+1}}^{j_{n+1}} \leq \Theta_{i_n}^{j_n}. \quad (4.2)$$

This establishes the existence part of the Lemma.

To establish $\xi_n \geq \xi_{n+1}$, we simply observe that ξ_n is at least as large as all the candidates that determine ξ_{n+1} . Indeed, $\xi_n = h_{j_n}(0) \geq h_{j_n-1}(0) = h_{j_{n+1}}(0)$. Next $\xi_n \geq \theta_{i_n}^l = \theta_{i_{n+1}}^l$ for $l \in \llbracket i_{n+1}, j_{n+1} - 1 \rrbracket$, and finally, $\xi_n > \Theta_{i_n}^{j_n} \geq \Theta_{i_{n+1}}^{j_{n+1}}$ by the right side inequality of (4.2). This completes the proof of the lemma. \blacksquare

LEMMA 4.2. *Suppose that in iteration n , the quantities $\theta_{i_n}^l, l \in \llbracket i_n, j_n - 1 \rrbracket$ and $\Theta_{i_n}^{j_n}$ exist. Suppose further that $\xi_n = \theta_{i_n}^t$ for some $t \in \llbracket i_n, j_n - 1 \rrbracket$, and Step 3(c) is executed. Then the following hold.*

- $y_m^* \in [0, \beta_m], m \in \llbracket i_n, t \rrbracket$.
- $\sum_{m=i_n}^l y_m^* \geq \sum_{m=i_n}^l \alpha_m, l \in \llbracket i_n, t \rrbracket$, with equality when $l = t$.
- The quantities $\theta_{i_{n+1}}^l, l \in \llbracket i_{n+1}, j_{n+1} - 1 \rrbracket$ and $\Theta_{i_{n+1}}^{j_{n+1}}$ exist.
- $\xi_n \geq \xi_{n+1}$.

Proof: Note that in this case t is chosen to be the largest one in $\llbracket i_n, j_n - 1 \rrbracket$ that satisfies $\xi_n = \theta_{i_n}^t$. Step 3(c) is executed; therefore $\theta_{i_n}^t \geq h_{j_n}(0) \geq h_m(0)$ for $m \in \llbracket i_n, t \rrbracket$. The assignment for y_m^* in the algorithm satisfies

$$y_m^* = H_m(\theta_{i_n}^t) \geq H_m(h_m(0)) = h_m^{-1}(h_m(0)) \wedge \beta_m = 0 \wedge \beta_m = 0.$$

That $y_m^* \leq \beta_m$ is obvious from the definition of H_m . This proves the upper and lower bound constraints on y_m^* .

To show that the ascending constraints (with the sum starting from i_n) hold for $l = \llbracket i_n, t \rrbracket$, observe that $\theta_{i_n}^t \geq \theta_{i_n}^l$ and the increasing property of H_m imply

$$\sum_{m=i_n}^l y_m^* = \sum_{m=i_n}^l H_m(\theta_{i_n}^t) \geq \sum_{m=i_n}^l H_m(\theta_{i_n}^l) = \sum_{m=i_n}^l \alpha_m,$$

with equality when $l = t$.

We next consider existence of $\theta_{i_{n+1}}^l$. Recall that in Step 3(c), $i_{n+1} = t + 1$ and $j_{n+1} = j_n$. Fix $l \in \llbracket i_{n+1}, j_{n+1} - 1 \rrbracket$. We simply set $\underline{\theta} = h_1(0)$ in (2.4). Moreover,

$$\begin{aligned} \sum_{m=i_{n+1}}^l \alpha_m &= \sum_{m=i_n}^l \alpha_m - \sum_{m=i_n}^t \alpha_m \\ &= \sum_{m=i_n}^l H_m(\theta_{i_n}^l) - \sum_{m=i_n}^t H_m(\theta_{i_n}^t) \quad (\text{from the definitions of } \theta_{i_n}^l \text{ and } \theta_{i_n}^t) \\ &\leq \sum_{m=i_n}^l H_m(\theta_{i_n}^l) - \sum_{m=i_n}^t H_m(\theta_{i_n}^l) \quad (\text{because } H_m \text{ is increasing}) \\ &= \sum_{m=i_{n+1}}^l H_m(\theta_{i_n}^l), \end{aligned}$$

and therefore we may set $\bar{\theta} = \theta_{i_n}^l$ in (2.4). $\theta_{i_{n+1}}^l$ therefore exists and

$$h_1(0) \leq \theta_{i_{n+1}}^l \leq \theta_{i_n}^l. \quad (4.3)$$

The same argument (mutatis mutandis to account for the sum of α_m up to K) establishes the existence of $\Theta_{i_{n+1}}^{j_{n+1}}$ and that

$$h_1(0) \leq \Theta_{i_{n+1}}^{j_{n+1}} \leq \Theta_{i_n}^{j_n}. \quad (4.4)$$

Finally, to show $\xi_n \geq \xi_{n+1}$, observe that $\xi_n \geq h_{j_n}(0) = h_{j_{n+1}}(0)$, $\xi_n \geq \Theta_{i_n}^{j_n} \geq \Theta_{i_{n+1}}^{j_{n+1}}$, and $\xi_n \geq \theta_{i_n}^l \geq \theta_{i_{n+1}}^l, l \in \llbracket i_{n+1}, j_{n+1} - 1 \rrbracket$. The last two facts follow from (4.4) and (4.3). So ξ_n is at least as large as all the candidates that determine ξ_{n+1} , i.e., $\xi_n \geq \xi_{n+1}$, and the proof is complete. \blacksquare

LEMMA 4.3. *Suppose that in iteration n , the quantities $\theta_{i_n}^l, l \in \llbracket i_n, j_n - 1 \rrbracket$ and $\Theta_{i_n}^{j_n}$ exist. Suppose further that $\xi_n = \Theta_{i_n}^{j_n}$. Then the following hold.*

- $y_m^* \in [0, \beta_m], m \in \llbracket i_n, j_n \rrbracket$.
- $\sum_{m=i_n}^l y_m^* \geq \sum_{m=i_n}^l \alpha_m, l \in \llbracket i_n, j_n \rrbracket$, with equality when $l = j_n$.

Proof: Under the hypotheses, Step 3(a) is executed. The proofs of the statements are identical to the proofs of the first two parts of Lemma 4.2 and is omitted. \blacksquare

PROPOSITION 4.4. *If $\theta_1^l, l \in \llbracket 1, L - 1 \rrbracket$ and Θ_1^L exist, the following statements hold.*

- *For every iteration step n with $i_n \leq j_n$, the quantities $\Theta_{i_n}^{j_n}$ and $\theta_{i_n}^l, l \in \llbracket i_n, j_n - 1 \rrbracket$ exist.*

- *Algorithm 1 terminates in $N \leq L$ steps.*
- *The output y^* of Algorithm 1 is feasible.*
- $\xi_1 \geq \xi_2 \geq \dots \geq \xi_N$.
- *In iteration N , Step 3(a) is executed.*

Proof: The key issue is the existence of $\theta_{i_n}^l$ and $\Theta_{i_n}^{j_n}$ in Step 3 of each iteration. The hypothesis of this Proposition resolves the issue for $n = 1$. Lemmas 4.1, 4.2, and 4.3 resolve the issue for subsequent iterations via induction. The first statement follows.

At least one variable is set in every iteration. The algorithm thus runs to completion in $N \leq L$ iterations, and the second statement holds.

The third and fourth statements also follow from Lemmas 4.1, 4.2, 4.3, and induction.

We now argue that Step 3(a) is executed in the last iteration. If this is not the case, the last iteration must be Step 3(b). This implies $i_N = j_N$ and $h_{j_N}(0) > \Theta_{j_N}^{j_N}$. The latter inequality and the definition of $\Theta_{j_N}^{j_N}$ yield

$$\sum_{m=j_N}^K \alpha_m = H_{j_N}(\Theta_{j_N}^{j_N}) \leq h_{j_N}^{-1}(\Theta_{j_N}^{j_N}) \leq h_{j_N}^{-1}(h_{j_N}(0)) = 0$$

contradicting our assumption (1.7) that $\sum_{m=L}^K \alpha_m > 0$. ■

4.2. Proof of Theorem 1. Proposition 4.4 implies the first two statements of Theorem 1. We now proceed to show the optimality of y^* to complete the proof of Theorem 1.

We use the Karush-Kuhn-Tucker (KKT) conditions (see for example [1, Sec. 3.3]) to show that the vector put out by the algorithm is a stationary point of a Lagrangian function with appropriately chosen Lagrange multipliers. The Lagrangian function for the problem is

$$\begin{aligned} & \sum_{m=1}^L g_m(y_m) + \sum_{m=1}^L \lambda_m^{(1)}(-y_m) + \sum_{m=1}^L \lambda_m^{(2)}(y_m - \beta_m) \\ & + \sum_{l=1}^{L-1} \lambda_l^{(3)} \left(-\sum_{m=1}^l y_m + \sum_{m=1}^l \alpha_m \right) + \mu \left(-\sum_{m=1}^L y_m + \sum_{m=1}^K \alpha_m \right) \end{aligned} \quad (4.5)$$

where $\lambda_m^{(1)}$ is the Lagrange multiplier that relaxes the positivity constraint $-y_m \leq 0$, $\lambda_m^{(2)}$ relaxes the upper bound constraint $y_m - \beta_m \leq 0$, $\lambda_m^{(3)}$ the ascending constraint (1.4), and μ the equality constraint (1.5). The KKT necessary and sufficient conditions for optimality of this convex optimization problem are given by:

$$\lambda_m^{(1)} y_m = 0, m = \llbracket 1, L \rrbracket \quad (4.6)$$

$$\lambda_m^{(2)} (y_m - \beta_m) = 0, m = \llbracket 1, L \rrbracket \quad (4.7)$$

$$\lambda_l^{(3)} \left(\sum_{m=1}^l y_m - \sum_{m=1}^l \alpha_m \right) = 0, l = \llbracket 1, L-1 \rrbracket \quad (4.8)$$

$$\lambda_m^{(1)} \geq 0, \lambda_m^{(2)} \geq 0, m = \llbracket 1, L \rrbracket, \text{ and } \lambda_l^{(3)} \geq 0, l = \llbracket 1, L-1 \rrbracket, \quad (4.9)$$

$$h_m(y_m) - \lambda_m^{(1)} + \lambda_m^{(2)} - \sum_{l=m}^{L-1} \lambda_l^{(3)} - \mu = 0, m = \llbracket 1, L \rrbracket. \quad (4.10)$$

Conditions (4.6), (4.7), and (4.8) are the complementary slackness conditions, (4.9) are the positivity conditions, and (4.10) identifies a stationary point for the Lagrangian function. We now choose appropriate values for the Lagrange multipliers and verify the KKT conditions.

First, let

$$\lambda_m^{(1)} = \begin{cases} \xi_{p_m} - \xi_N, & \text{if } c_m = \mathcal{B}^*, \\ 0, & \text{otherwise.} \end{cases} \quad (4.11)$$

Recall that p_m is the iteration number in which variable y_m was set, and that $c_m = \mathcal{B}^*$ whenever Step 3(b) is executed, i.e., $y_m^* = 0$. From the assignment in (4.11), $\lambda_m^{(1)} \neq 0$ implies that $c_m = \mathcal{B}^*$ and therefore $y_m^* = 0$. Thus the complementary slackness condition (4.6) is satisfied for $m = \llbracket 1, L \rrbracket$.

Second, let

$$\lambda_m^{(2)} = \begin{cases} 0, & \text{if } c_m = \mathcal{B}^*, \\ \xi_{p_m} - h_m(H_m(\xi_{p_m})), & \text{otherwise.} \end{cases} \quad (4.12)$$

If $\lambda_m^{(2)} \neq 0$, then from (4.12) we have $\xi_{p_m} \neq h_m(H_m(\xi_{p_m}))$. From the strictly increasing property of h_m and the definition of H_m , we have

$$h_m(H_m(\xi_{p_m})) = h_m(h_m^{-1}(\xi_{p_m}) \wedge \beta_m) = h_m(h_m^{-1}(\xi_{p_m})) \wedge h_m(\beta_m) = \xi_{p_m} \wedge h_m(\beta_m), \quad (4.13)$$

so that $h_m(H_m(\xi_{p_m})) \neq \xi_{p_m}$ implies that $H_m(\xi_{p_m})$ must have saturated to β_m , i.e., $y_m^* = H_m(\xi_{p_m}) = \beta_m$. The complementary slackness conditions (4.7) are therefore fulfilled.

Third, for $l = \llbracket 1, L-1 \rrbracket$ let

$$\lambda_l^{(3)} = \begin{cases} \xi_{p_l} - \xi_{p_k}, & \text{if } c_l = \mathcal{C}^*, \\ 0, & \text{otherwise,} \end{cases} \quad (4.14)$$

where

$$p_k := \min \{p_m : m \in \llbracket 1, L \rrbracket, p_m > p_l, c_m \in \{\mathcal{C}^*, \mathcal{A}^*\}\}. \quad (4.15)$$

The last iteration is always via Step 3(a) (Proposition 4.4). Thus, when $c_l = \mathcal{C}^*$, there is a later iteration that executes Step 3(a) which implies that the set in (4.15) is nonempty and that the assignment (4.14) is well-defined. Suppose $\lambda_l^{(3)} \neq 0$. Then $c_l = \mathcal{C}^*$, an asterisked assignment. The second statement of Lemma 4.2 therefore ensures that the ascending constraint is satisfied with equality for this l . The complementary slackness condition (4.8) is thus fulfilled for $l = \llbracket 1, L-1 \rrbracket$.

The assignment of $\lambda_l^{(3)}$ in (4.14) can be equivalently expressed as

$$\lambda_l^{(3)} = \begin{cases} \sum_{n=p_l}^{p_k-1} \xi_n - \xi_{n+1}, & \text{if } c_l = \mathcal{C}^*, \\ 0, & \text{otherwise,} \end{cases} \quad (4.16)$$

where p_k is given by (4.15). This will be useful in verifying (4.10).

Finally, we set $\mu = \xi_N$.

The Lagrange multiplier assignments in (4.11), (4.12), and (4.14) are positive. Indeed, the positivity in (4.11) and (4.14) follow from the monotonicity property $\xi_n \geq \xi_{n+1}, n = \llbracket 1, N-1 \rrbracket$ (Proposition 4.4). The positivity of $\lambda_m^{(2)}$ follows from

$$h_m(H_m(\xi_{p_m})) \leq h_m(h_m^{-1}(\xi_{p_m})) = \xi_{p_m}.$$

All that remains is to verify (4.10). To do this, first consider $m > j_N$. Then the assignments $y_m^* = 0$ and $y_l^* = 0, l = \llbracket m+1, L-1 \rrbracket$ are via Step 3(b); therefore $\xi_{p_l} = h_l(0)$ and $c_l = \mathcal{B}^*$. The latter implies $\lambda_m^{(1)} = \xi_{p_m} - \xi_N$, $\lambda_m^{(2)} = 0$, and $\lambda_l^{(3)} = 0$ for $l = \llbracket m, L-1 \rrbracket$. Substitution of these assignments in (4.10) yields

$$h_m(0) - \lambda_m^{(1)} + \lambda_m^{(2)} - \sum_{l=m}^{L-1} \lambda_l^{(3)} - \mu = \xi_{p_m} - (\xi_{p_m} - \xi_N) + 0 - 0 - \xi_N = 0.$$

Now consider $m \in \llbracket 1, j_N \rrbracket$ and $p_m < N$, i.e., variable y_m is not set in the last iteration. Then $c_m \in \{\mathcal{C}, \mathcal{C}^*\}$, and thus $c_m \neq \mathcal{B}^*$. Substitution of (4.11), (4.12), and (4.14) in (4.10) yields

$$\begin{aligned} h_m(y_m^*) - \lambda_m^{(1)} + \lambda_m^{(2)} - \sum_{l=m}^{L-1} \lambda_l^{(3)} - \mu \\ = h_m(H_m(\xi_{p_m})) - 0 + (\xi_{p_m} - h_m(H_m(\xi_{p_m}))) - \sum_{l=m}^{L-1} \lambda_l^{(3)} - \xi_N \\ = \xi_{p_m} - \xi_N - \sum_{l=m}^{L-1} \lambda_l^{(3)} \end{aligned} \quad (4.17)$$

$$\begin{aligned} &= \xi_{p_m} - \xi_N - \sum_{n=p_m}^{N-1} (\xi_n - \xi_{n+1}) \\ &= \xi_{p_m} - \xi_N - (\xi_{p_m} - \xi_N) \\ &= 0. \end{aligned} \quad (4.18)$$

In the above sequence of inequalities, (4.18) holds because of the following. In (4.17), the summation over l has only one nonzero entry per iteration, i.e., whenever $c_l = \mathcal{C}^*$. We may therefore sum over the iteration index n instead of the variable index l . Iterations p_m to $N-1$ involve the execution of either Step 3(b) or Step 3(c). Substitution of (4.16) in (4.17) then results in (4.18).

Lastly, consider $m \in \llbracket 1, j_N \rrbracket$ and $p_m = N$, i.e., y_m is assigned in the last iteration. From Proposition 4.4, Step 3(a) is executed in this iteration, and therefore $c_m \in \{\mathcal{A}, \mathcal{A}^*\}$. Then

$$\begin{aligned} h_m(y_m) - \lambda_m^{(1)} + \lambda_m^{(2)} - \sum_{l=m}^{L-1} \lambda_l^{(3)} - \mu \\ = h_m(H_m(\xi_{p_m})) - 0 + (\xi_N - h_m(H_m(\xi_{p_m}))) - 0 - \xi_N = 0. \end{aligned}$$

The output y^* of Algorithm 1 and the Lagrange multiplier assignments satisfy the KKT conditions; y^* therefore minimizes (1.2), and the proof of Theorem 1 is complete. \blacksquare

4.3. Proof of Corollary 2.1. The assignments in Algorithm 1 are

$$y_m^* = H_m(\xi_{p_m}), m = \llbracket 1, L \rrbracket.$$

By hypothesis, $H_m \geq H_{m+1}, m \in \llbracket 1, L-1 \rrbracket$, and by Proposition 4.4, $\xi_n \geq \xi_{n+1}, n \in \llbracket 1, N-1 \rrbracket$. These monotonicity properties imply

$$y_m^* = H_m(\xi_{p_m}) \geq H_{m+1}(\xi_{p_m}) \geq H_{m+1}(\xi_{p_{m+1}}) = y_{m+1}^*, m \in \llbracket 1, L-1 \rrbracket. \quad \blacksquare$$

4.4. Proof of Proposition 2.2. : Recall that here $K = L$. Define

$$\mathcal{L}(\alpha) := \{y \in \mathbb{R}^L : y \text{ satisfies (1.3) -- (1.5)}\}.$$

$\mathcal{L}(\alpha)$ is convex, but may not be closed because the domains (a_m, b_m) may not be closed. From the ascending constraints (1.4) and (1.5), it is clear that if $\alpha \succeq \tilde{\alpha}$ then $\mathcal{L}(\alpha) \subseteq \mathcal{L}(\tilde{\alpha})$, and therefore $\mathcal{G}(\alpha) \geq \mathcal{G}(\tilde{\alpha})$. The first statement is therefore proved. (Note that the conditions on $g_m, m \in \llbracket 1, L \rrbracket$ stated in Section 1 are not necessary for this property).

To show convexity, consider $\alpha, \tilde{\alpha} \in \mathbb{R}_+^L$. Fix $\lambda \in (0, 1)$. If either of $\mathcal{L}(\alpha)$ or $\mathcal{L}(\tilde{\alpha})$ is empty, there is nothing to prove. We may therefore assume both are nonempty and therefore $\mathcal{G}(\alpha)$ and $\mathcal{G}(\tilde{\alpha})$ are finite. For every $\varepsilon > 0$, there exist $y \in \mathcal{L}(\alpha)$ and $\tilde{y} \in \mathcal{L}(\tilde{\alpha})$ satisfying $G(y) < \mathcal{G}(\alpha) + \varepsilon$ and $G(\tilde{y}) < \mathcal{G}(\tilde{\alpha}) + \varepsilon$. The linearity of the constraints implies $\lambda y + (1 - \lambda)\tilde{y} \in \mathcal{L}(\lambda\alpha + (1 - \lambda)\tilde{\alpha})$. The convexity of G implies

$$\begin{aligned} \mathcal{G}(\lambda\alpha + (1 - \lambda)\tilde{\alpha}) &\leq G(\lambda y + (1 - \lambda)\tilde{y}) \\ &\leq \lambda G(y) + (1 - \lambda)G(\tilde{y}) \\ &\leq \lambda \mathcal{G}(\alpha) + (1 - \lambda)\mathcal{G}(\tilde{\alpha}) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, the convexity of \mathcal{G} is established. ■

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